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Natural vibration of two-dimensional slender structure–water interaction systems subject to Sommerfeld radiation condition

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Abstract

The dynamic behaviour of two-dimensional flexible slender structure–water interaction systems subject to a Sommerfeld radiation condition at the infinity boundary of the water domain is investigated. A new parameter, the *speed of radiation wave*, is introduced into the Sommerfeld radiation condition to consider the influences of both of the pressure wave and the free surface wave of the water, which is an extension of the original Sommerfeld condition. The governing equations describing the dynamic behaviour of the system are analysed and solved using a separation of variables method. It is demonstrated that the natural vibration of the two-dimensional slender structure–water interaction system subject to a Sommerfeld radiation condition is governed by a complex eigenvalue equation which has only pairs of complex conjugate eigenvalues. The number of the pairs of complex conjugate natural frequencies equals the number of the natural modes of the corresponding dry structure and is independent of the continuous fluid domain, which has infinite degrees of freedom. The examples, including four cases of shallow water, deep water, no free surface wave and incompressible water, demonstrate and illustrate the developed theoretical and numerical method.

1. Introduction

Sommerfeld's original proof [1] of the uniqueness theorem of the radiation solution $\varphi = \phi e^{-i\omega t}$ of the wave equation defined in a full infinite three-dimensional space assumed an additional condition

$$\lim_{r\to\infty} r\left(\frac{\partial\phi}{\partial r} - i\kappa\phi\right) = 0, \quad \kappa = \omega/c. \tag{1}$$

Here, the quantity r stands for the distance from any fixed point $r = 0$, $i = \sqrt{-1}$, κ represents the ratio of the circular frequency ω of the stimulation and c denotes the speed of wave in a full infinite three-dimensional space. This condition is called the general condition of radiation [2]. The fact that this condition is superfluous has been rigorously proven by Rellich [3] even for the case of an arbitrary number of dimensions h where the

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radiation condition reads [4]

$$\lim_{r\to\infty} r^{(h-1)/2}\left(\frac{\partial\phi}{\partial r} - \mathrm{i}\kappa\phi\right) = 0. \tag{2}$$

The Sommerfeld condition has been widely adopted to investigate wave radiation problems, for example, see, Refs. [5–7]. Moreover, Filippi [8] used a Fourier transformation method to derive the dynamic response of a one-dimensional vibro-acoustic system excited by an external force and subjected to the imposed Sommerfeld radiation condition at infinity.

It is well known (see, for example, Refs. [4,9]) that the natural vibration of a dynamic system is defined by an eigenvalue problem of the corresponding idealised system with no *material* damping assumed and external forces. From the defined eigenvalue problem, the real natural frequencies and modes of the system are theoretically derived or numerically calculated using finite element methods [10,11]. For example, Morand and Ohayon [12] presented some detailed methods for numerical modelling of linear natural vibration analysis of elastic structures coupled to internal fluids. Xing and Price [13] and Xing et al. [14] proposed a mixed finite element substructure-subdomain method to simulate natural vibrations and dynamic responses of various linear fluid–structure interaction problems.

Following the definition of the natural vibration of a system, Xing et al. [15] investigated the natural vibrations of a beam–water interaction system subject to a non-disturbance condition at infinity boundary of the water. Zhao et al. [16] further studied this beam–water interaction system with a concentrated mass and moment of inertia added at the top of beam. In these two papers, more publications involving beam–water interactions are cited and therefore a repeating review is omitted in this paper. However, we may ask two questions: what are the natural dynamic characteristics of a beam–water interaction system subject to a Sommerfeld radiation condition at an infinity boundary? What is a suitable radiation condition for the system involving both of the free surface wave on the free surface boundary of the water and the pressure wave in the water domain? The general solution of these problems has not been reported, although Xing [17] presented limited simple examples to draw some fundamental concepts of the problem and Zhao [18] numerically calculated a beam–water system involving the Sommerfeld radiation condition using the method reported in Ref. [15]. This paper continues these researches to address this fundamental problem by investigating a two-dimensional slender structure (or beamlike structure)–water interaction system using a theoretical analysis in association with the validations by examples.

Before starting our discussion, it may be useful for readers to note that the system studied in this paper is a natural vibration system to which there are no external/internal forces or vibration/wave sources applied and also no material damping in the structure and the water. This is different from aforementioned publications which studied various radiation problems in which at least one vibration/wave source exists.

2. Governing equations

Fig. 1 illustrates a two-dimensional flexible slender structure–water interacting system in which a concentrated mass m_0 with moment I_0 of inertia is attached to the free end of the slender structure coupled to a water domain $0 \leqslant x \leqslant \infty$, $0 \leqslant y \leqslant h$. The slender structure width F is assumed negligibly small compared to the infinite fluid domain and the length of the structure. Here, $o-xy$ represents a two-dimensional Cartesian coordinate system with its origin o located at the intersection of the central line of the slender structure and the horizontal floor of the reservoir. We consider that the water of mass density ρ_f is compressible, inviscid and its motion irrotational. The flexible uniform slender structure is of bending stiffness EJ, mass density ρ_s and thickness $B = 1$ in the $o-z$ direction perpendicular to the $o-xy$ plane.

For engineering applications and further research considerations intended by interested readers, it is necessary to explain the details of this system. The system may be considered as a model of a dam–water interaction system in which the dam with its top mass and water domain are assumed to be infinitely long in the $o-z$ direction. Therefore the strain of the dam in $o-z$ direction vanishes which constructs a classical plane strain problem from which we assume that the water motion has the same pattern in all planes parallel to the $o-xy$ plane to match the dam deformation. As a result of this, the system can be analysed using a two-dimensional sheet of the dam and the water of thickness $B = 1$. Further more, as normally used in

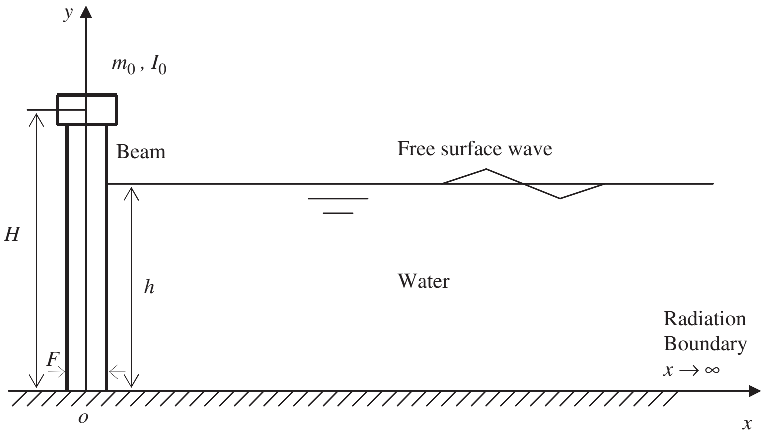

Fig. 1. A two-dimensional slender structure–water interaction system subject to a Sommerfeld radiation condition at infinite boundary $x \to \infty$.

engineering, the height H of the dam is significantly larger than its width F and thickness B, so that the sheet of the dam is considered as a slender structure and the classical beam theory is valid to describe the deformation of this sheet of the dam that implies that only a deflection in $o-x$ direction is considered as a variable in the analysis of the deformation of the dam sheet.

Under the assumption of small disturbances, the linearised equations describing the dynamic pressure $p(x, y, t)$ in the water, the horizontal deflection $u(y, t)$, $0 \leqslant y \leqslant H$, of the slender structure are as follows.

2.1. Governing equations

2.1.1. Fluid domain

Dynamic equation:

$$\partial^2 p/\partial x^2 + \partial^2 p/\partial y^2 = (1/c^2)\partial^2 p/\partial t^2, \quad 0 < x < \infty, \ 0 < y < h, \tag{3}$$

where c denotes the speed of sound in the water.

Boundary conditions:

On the free surface, a free surface wave is considered

$$\partial p/\partial y = -(1/g)\partial^2 p/\partial t^2, \quad y = h, \tag{4}$$

where g represents the acceleration due to gravity.

On the bottom of the reservoir, assumed impermeable and rigid,

$$\partial p/\partial y = 0, \quad y = 0. \tag{5}$$

At infinity in the water domain, it is assumed that a radiation condition in x-direction applies, i.e.

$$\lim_{x \to \infty} (\partial p/\partial x - \mathrm{i}kp) = 0, \tag{6}$$

where a parameter k, which is different from the notation κ used in Eqs. (1) and (2), is introduced. In this paper, we consider only the natural vibration solutions of the system. Therefore the pressure in the water has the form $p(x, y, t) = P(x, y)\mathrm{e}^{-\mathrm{i}\omega t}$ so that we can write Eq. (6) in an equivalent form

$$\lim_{x \to \infty} \left(\partial p/\partial x + \frac{1}{\upsilon}\partial p/\partial t \right) = 0, \tag{7}$$

where $\upsilon = \omega/k$ is introduced as a *real speed of the radiation wave* to consider the influences of the boundary conditions of the problem. This speed of the radiation wave will equal the speed c of sound in a full infinite three, two or one-dimensional space where only the pressure wave is considered but no free surface waves involved as investigated by Sommerfeld [1,2].

2.1.2. Solid domain

Dynamic equation:

$$EJ(\partial^4 u/\partial y^4) + \rho_s FB(\partial^2 u/\partial t^2) = -\beta(y)p(0,y,t)B, \quad 0<y<H, \tag{8}$$

$$\beta(y) = \begin{cases} 1, & 0\leqslant y\leqslant h, \\ 0, & h<y\leqslant H. \end{cases} \tag{9}$$

Boundary conditions:

At the base of the structure, assumed to be fixed,

$$u=0, \quad \partial u/\partial y = 0, \quad y=0. \tag{10}$$

At the free end, considering the concentrated mass m_0 with moment I_0 of inertia, we derive the following boundary conditions:

$$EJ(\partial^2 u/\partial y^2) = -I_0[\partial^3 u/(\partial t^2 \partial y)], \quad EJ(\partial^3 u/\partial y^3) = m_0\, \partial^2 u/\partial t^2, \quad y=H. \tag{11}$$

2.1.3. Fluid– structure interaction interface

On the fluid–structure interaction interface, we assume an impermeable and motion consistent boundary condition, which implies that the fluid cannot flow into the structure and has the same displacement, velocity and acceleration in $o-x$ direction as on the wet interface of the structure. Therefore, the pressure p in the water and the horizontal displacement u of the wet structure section satisfy Eq. (8) and the relation [15]

$$\partial p/\partial x = -\rho_f(\partial^2 u/\partial t^2), \quad x=0, \quad 0<y<h. \tag{12}$$

2.2. Variable separable forms of governing equations

By using the separation of variables method (see, for example, Ref. [4]), solutions of the pressure p, displacement u for natural vibrations of the system are sought in the forms

$$p(x,y,t) = P(x,y)T(t) = X(x)Y(y)T(t) = X(x)Y(y)\mathrm{e}^{-\mathrm{i}\omega t}, \tag{13}$$

$$u(y,t) = U(y)T(t) = U(y)\mathrm{e}^{-\mathrm{i}\omega t}, \tag{14}$$

where ω denotes the square root of an eigenvalue of the system or a natural frequency of the system.

The substitution of these expressions into Eqs. (3)–(12) allows separation of variables and the following sets of equations are obtained.

Spatial y-function $Y(y)$:

$$Y'' + \lambda^2 Y = 0, \quad Y'(0) = 0, \quad Y'(h) = -\omega^2 Y(h)/g. \tag{15}$$

Spatial x-function $X(x)$:

$$X'' + k^2 X = 0, \quad 0<x<\infty, \quad X'(\infty) - \mathrm{i}kX(\infty) = 0. \tag{16}$$

Displacement function $U(y)$:

$$EJU^{(4)} - \rho_s FB\omega^2 U = -P(x,y)\beta(y)B = -X(0)Y(y)\beta(y)B, \quad 0<y<H, \tag{17}$$

$$U_1(0) = 0, \quad U_1'(0) = 0, \tag{18}$$

$$EJU'''(H) = \omega^2 I_0 U'(H), \quad EJU'''(H) = -\omega^2 m_0 U(H), \tag{19}$$

Fluid–solid interaction condition:

$$(\partial P/\partial x)(0,y) = X'(0)Y(y) = \rho_f \omega^2 U(y), \quad 0<y<h. \tag{20}$$

Here an apostrophe implies a spatial differentiation and the parameters λ^2 and k^2 satisfy the relation

$$\lambda^2 + k^2 = \omega^2/c^2. \tag{21}$$

3. Eigenvalue equations

The function Y satisfying Eq. (15) takes the form [15]

$$Y_n(y) = \cos(\lambda_n y), \quad n = 1,2,3\ldots \tag{22}$$

where λ_n are the solutions of equation

$$\lambda_n \tan(\lambda_n h) = -\frac{\omega^2}{g}. \tag{23}$$

As described in Ref. [15], for a natural frequency ω, there exist a series of solutions, identified by subscript n, of Eq. (23), which should be considered in the following derivation.The functions $Y_n(y)$ satisfy the following orthogonal relationship:

$$\int_0^h Y_m Y_n \, dy = \begin{cases} 0, & m \neq n, \\ \dfrac{2\lambda_n h + \sin(2\lambda_n h)}{4\lambda_n}, & m = n, \end{cases} \tag{24}$$

and for a natural frequency ω, the solutions of Eq. (23) and the corresponding parameters k_n must satisfy Eq. (21), i.e.

$$\frac{\omega^2}{c^2} = k_n^2 + \lambda_n^2, \quad n = 1,2,3,\ldots \tag{25}$$

The function X_n satisfying Eq. (16) takes the form

$$X_n(x) = e^{ik_n x}. \tag{26}$$

The pressure in the fluid is now expressed as a summation in the form

$$p = P(x,y)e^{-i\omega t} = \sum_{n=1,2,\ldots} P_n e^{ik_n x} \cos(\lambda_n y) e^{-i\omega t}, \tag{27}$$

where P_n are constants to be determined.

For a more general discussion of the proposed solution, the following non-dimensional parameters are defined:

$$\begin{aligned} &\xi = y/H, \quad v = h/H, \quad \gamma = \rho_f/\rho_s, \quad \bar{\omega} = \omega/\omega_b, \quad \bar{c} = c/(\omega_b H), \quad \bar{k} = kH, \\ &\bar{\lambda} = \lambda H, \quad \bar{\lambda}^2 + \bar{k}^2 = \bar{\omega}^2/\bar{c}^2, \quad r_m = m_0/(\rho_s FHB), \quad r_I = I_0/(m_0 H^2). \end{aligned} \tag{28}$$

Here $\omega_b = \sqrt{EJ/(\rho_s FBH^4)}$ represents the frequency parameter of the dry structure. The variable ξ denotes a non-dimensional coordinate in y direction and v defines the ratio of water depth to structure length; γ represents the ratio of the mass density of water to the one of the structure and $\bar{\omega}$ denotes the non-dimensional frequency.

Substituting Eqs. (27) and (28) into Eqs. (17)–(20), we derive

$$\bar{U}^{(4)}(\xi) - \bar{\omega}^2 \bar{U}(\xi) = -\beta(\xi) \sum_{n=1,2,3,\ldots} \bar{P}_n \cos(\bar{\lambda}_n \xi), \quad 0<\xi<1, \tag{29}$$

$$\beta(\xi) = \begin{cases} 1, & 0 \leqslant \xi \leqslant v, \\ 0, & v < \xi \leqslant 1. \end{cases} \tag{30}$$

$$\bar{\omega}^2 \bar{U}(\xi) = \sum_{n=1,2,3,\ldots} \mathrm{i}(\bar{k}_n/\gamma)\bar{P}_n \cos(\bar{\lambda}_n \xi), \quad 0 < \xi < v, \tag{31}$$

$$\bar{U}(0) = 0 = \bar{U}'(0), \quad \bar{U}''(1) = \bar{\omega}^2 \bar{I}_0 \bar{U}'(1), \quad \bar{U}'''(1) = -\bar{\omega}^2 \bar{m}_0 \bar{U}(1). \tag{32}$$

Here $\bar{P}_n = P_n/[EJ/(FBH^2)]$, $\bar{I}_0 = \omega_b^2 I_0 H/(EJ)$ and $\bar{m}_0 = \omega_b^2 m_0 H^3/(EJ)$ are non-dimensional parameters. The corresponding non-dimensional forms of Eqs. (23) and (24) are

$$\bar{\lambda}_n \tan(\bar{\lambda}_n v) = -\frac{\bar{\omega}^2}{\bar{g}}, \quad \bar{g} = g/(H\omega_b^2),$$
$$\int_0^v \bar{Y}_m(\xi)\bar{Y}_n(\xi)\,\mathrm{d}\xi = \begin{cases} 0, & m \neq n, \\ \dfrac{v}{2} + \dfrac{\sin(2\bar{\lambda}_n v)}{4\bar{\lambda}_n} = \Lambda_n, & m = n. \end{cases} \tag{33}$$

Since the natural modes of the dry structure are a set of orthogonal and complete base vectors forming a mode space to represent any motions of the structure [4], therefore, in using the mode summation method based on the natural modes of the dry structure, the solution $\bar{U}(\xi)$ is expressed in the form

$$\bar{U}(\xi) = \mathbf{\Phi}\mathbf{Q}, \quad \mathbf{\Phi} = [\,\phi_1(\xi) \quad \phi_2(\xi) \quad \cdots \quad \phi_N(\xi)\,], \quad \mathbf{Q}^{\mathrm{T}} = [\,Q_1 \quad Q_2 \quad \cdots \quad Q_2\,], \tag{34}$$

where $\mathbf{\Phi}$ denotes a line vector consisting of the N retained non-dimensional dry modes $\phi_J(\xi)$ ($J = 1, 2, 3, \ldots, N$), of the structure and $\mathbf{Q}$ represents a column vector of generalised coordinates. The dry modes of the structure satisfy the boundary conditions (32) and the following orthogonal relations:

$$\int_0^1 \phi_I(\xi)\phi_J(\xi)\,\mathrm{d}\xi + \phi_I(1)\bar{m}_0\phi_J(1) + \phi_I'(1)\bar{I}_0\phi_J'(1) = \begin{cases} 1, & I = J, \\ 0, & I \neq J, \end{cases} \tag{35}$$

and

$$\int_0^1 \phi_I''(\xi)\phi_J''(\xi)\,\mathrm{d}\xi = \begin{cases} \Omega_I^2, & I = J, \\ 0, & I \neq J. \end{cases} \tag{36}$$

Here, Ω_I represents the Ith non-dimensional natural frequency of the dry structure. Substituting Eq. (34) into Eq. (29), and then pre-multiplying by $\mathbf{\Phi}^{\mathrm{T}}$, integrating the resultant equation with respect to ξ from 0 to 1 and using Eqs. (35) and (36), we obtain a matrix equation

$$\mathrm{diag}(\Omega_I^2 - \bar{\omega}^2)\mathbf{Q} + \mathbf{\Psi}\bar{\mathbf{P}} = \mathbf{0}, \tag{37}$$

where

$$\bar{\mathbf{P}}^{\mathrm{T}} = [\,\bar{P}_1 \quad \bar{P}_2 \quad \cdots \quad \bar{P}_M\,]. \tag{38}$$

Here we assume that the M successive solutions $\bar{\lambda}_n$ ($n = 1, 2, 3, \ldots, M$), of Eq. (33) are retained and therefore $\mathbf{\Psi}$ is a $N \times M$ matrix of which a representative element

$$\Psi_{IJ} = \int_0^v \phi_I(\xi)\cos(\bar{\lambda}_J \xi)\,\mathrm{d}\xi. \tag{39}$$

Pre-multiplying Eq. (31) by $[\,\cos(\bar{\lambda}_1\xi) \quad \cos(\bar{\lambda}_2\xi) \quad \cdots \quad \cos(\bar{\lambda}_M\xi)\,]^{\mathrm{T}}$, and then integrating with respect to ξ from 0 to v and using Eq. (33), we obtain another matrix form

$$\mathbf{\Psi}^{\mathrm{T}}\mathbf{Q} - \frac{\mathrm{i}}{\bar{\omega}^2\gamma}\mathrm{diag}(\Lambda_n \bar{k}_n)\bar{\mathbf{P}} = \mathbf{0}. \tag{40}$$

Combining Eqs. (37) and (40), we have the following matrix equation:

$$\begin{bmatrix} \mathrm{diag}(\Omega_I^2 - \bar{\omega}^2) & \mathbf{\Psi} \\ \bar{\omega}^2 \mathbf{\Psi}^{\mathrm{T}} & -(\mathrm{i}/\gamma)\mathrm{diag}(\Lambda_n \bar{k}_n) \end{bmatrix} \begin{bmatrix} \mathbf{Q} \\ \bar{\mathbf{P}} \end{bmatrix} = \mathbf{0}. \tag{41}$$

The necessary and sufficient condition for Eq. (41) to have a non-trivial solution is that its determinant of the coefficient matrix vanishes, i.e.

$$\begin{vmatrix} \mathrm{diag}(\Omega_I^2 - \bar{\omega}^2) & \mathbf{\Psi} \\ \bar{\omega}^2 \mathbf{\Psi}^{\mathrm{T}} & -(\mathrm{i}/\gamma)\mathrm{diag}(\Lambda_n \bar{k}_n) \end{vmatrix} = \bar{\omega}^{2N} \begin{vmatrix} \mathrm{diag}(\Omega_I^2/\bar{\omega}^2 - \mathbf{I}_{N\times N}) & \mathbf{\Psi} \\ \mathbf{\Psi}^{\mathrm{T}} & -(\mathrm{i}/\gamma)\mathrm{diag}(\Lambda_n \bar{k}_n) \end{vmatrix} = \mathbf{0}. \tag{42}$$

Eq. (42) gives the characteristic equation of the structure–water interaction system. From this equation, the natural frequency $\bar{\omega}$ can be determined and then the corresponding natural mode is derived from Eqs. (41) and (34).

To find a solution of Eq. (42), a numerical iteration process is necessary. The calculation process is as follows:

(i) Determine the region of each natural frequency of the system according to the natural frequencies of the dry structure and the solution characteristics described in Section 4.
(ii) Starting from a trial solution $\tilde{\bar{\omega}}$, we can find the corresponding parameters $\bar{\lambda}_n$, Λ_n, $\bar{k}_n$ and Ψ_{IJ} by solving Eq. (33) and using Eqs. (28) and (37)–(39).
(iii) Solve Eq. (42) to find an approximate solution $\bar{\omega}$.
(iv) Check if a convergence condition of $\left|\bar{\omega} - \tilde{\bar{\omega}}\right| / |\bar{\omega}| \leqslant \varepsilon$ is reached. Here ε is a small error allowed. If the convergence is not reached, return to (ii) using the obtained approximate solution $\bar{\omega}$ as a new starting value of the solution.

A similar numerical iteration process has been used and demonstrated in our previous publications for the beam–water interaction analysis [15,16,18]. This paper does not intend to provide a detailed numerical process but to highlight an analysis of the new characteristics of the natural vibrations of the system caused by the Sommerfeld radiation condition. To validate the analysis using examples, four cases will be studied.

4. Analysis of solution characteristics

Here, we analyse the characteristics of the natural frequencies of the structure–water interaction system subject to the Sommerfeld radiation condition.

4.1. A complex identity

Since the governing equations of the problem are a set of equations with only real coefficients, if a solution consisting of displacement u, pressure p and natural frequency ω satisfies Eqs. (3)–(12), the solution consisting of the conjugate functions u^*, p^* and ω^* of this solution must also be a solution of the conjugate equations [4]. Therefore, the conjugate functions of the solutions given in Eqs. (13) and (14) are a set of functions:

$$\begin{gathered} p^*(x,y,t) = P^*(x,y)T^*(t) = X^*(x)Y^*(y)\mathrm{e}^{\mathrm{i}\omega^* t}, \\ u_1^*(y,t) = U_1^*(y)\mathrm{e}^{\mathrm{i}\omega^* t}, \quad u_2^*(y,t) = U_2^*(y)\mathrm{e}^{\mathrm{i}\omega^* t}, \end{gathered} \tag{43}$$

providing a conjugate solution of the conjugate equations of Eqs. (3)–(12). Using the same method to derive the orthogonal relation of the natural vibration forms of the structure–water interaction system subject to an undisturbed condition at the infinite boundary $x \to \infty$ of the water [15], we obtain a complex relation for the two sets of solutions, identified by subscripts m and n, of the structure–water interaction system studied herein as follows:

$$(\omega_m^* - \omega_n)\left\{(1/c^2)\int_{\Omega_f} P_n P_m^* \,\mathrm{d}\Omega + EJ\rho_f\left[\int_0^h U_{1n}'' U_{1m}^{*}{}'' \,\mathrm{d}y + \int_0^h U_{2n}'' U_{2m}^{*}{}'' \,\mathrm{d}y\right] + (1/g)\int_0^\infty P_n(x,h)P_m^*(x,h)\,\mathrm{d}x\right\}$$
$$= \mathrm{i}(1/\upsilon)\lim_{x\to\infty}\int_0^h P_n(x,y)P_m^*(x,y)\,\mathrm{d}y. \tag{44}$$

This relation is used to analyse the characteristics of the natural vibrations of the structure–water interaction system.

4.2. *Characteristics of the solution*

4.2.1. *Complex natural frequency with a negative imaginary part*

For a same natural mode, i.e. $m = n$, Eq. (44) reduces to

$$(\omega_n^* - \omega_n)\left\{(1/c^2)\int_{\Omega_f} |P_n|^2 \,\mathrm{d}\Omega + EJ\rho_f\left[\int_0^h \left|U_{1n}''\right|^2 \mathrm{d}y + \int_0^h \left|U_{2n}''\right|^2 \mathrm{d}y\right] + (1/g)\int_0^\infty \left|P_n(x,h)\right|^2 \mathrm{d}x\right\}$$
$$= \mathrm{i}(1/\upsilon)\lim_{x\to\infty}\int_0^h \left|P_n(x,y)\right|^2 \mathrm{d}y. \tag{45}$$

From this equation it follows that:

(i) the natural frequency of the system must be complex, otherwise, Eq. (45) does not hold since for a real frequency $\omega_n^* = \omega_n$ the left side of Eq. (45) vanishes but its right side $\int_0^h \left|P_n(x,y)\right|^2 > 0$ required by the radiation condition;
(ii) the natural frequency ω has a negative imaginary part since $\omega_n^* - \omega_n = -\mathrm{i}2\mathrm{Im}(\omega_n)$ must be positive for Eq. (45) to be valid.

This finding reveals that a natural vibration of the structure–water interaction system subject to the Sommerfeld radiation condition behaves as a damped vibration with an energy dissipation process although there is no material damping in both the fluid and the solid, which is a natural characteristic of the system and does not depend on whether or not there exists any vibration source or force.

4.2.2. *Number of the natural frequencies of the system is independent of the infinite water domain*

It is observed that the highest power of $\bar{\omega}^2$ in Eq. (42) is equal to the number of the retained dry modes of the structure, i.e. N, which is independent of the infinite fluid. Therefore, the number of complex conjugate natural frequencies of the system equals the number of degrees of freedom of the dry structure.

This finding implies that for the structure–water interaction system subject to a Sommerfeld radiation condition at the infinite boundary of the water there are no extra numbers of natural vibrations except the numbers of natural vibrations of the dry structure. This characteristic is much different from the one for the beam–water interaction system subject to a non-disturbance boundary condition on the infinite boundary of the water, of which the number of natural vibrations depends on both of the dry beam and the water domain [15].

Based on this characteristic, it is sufficient to choose the natural modes of the dry structure and use a mode summation method to calculate dynamic responses of the structure–water interaction system subject to a Sommerfeld radiation condition since the natural modes of the dry structure constructs a complete orthogonal space to describe any motions of the structure [4,9].

5. Case studies

To illustrate and to validate the developed method described previously, we investigate the following four cases. These cases can avoid numerically complex iteration calculations but clearly reveal the physical mechanisms of the system caused by the Sommerfeld radiation condition.

5.1. Shallow water

We assume that the depth h of the water domain is very small, i.e. a shallow water case. For this case, we have an approximation $\tan(\bar{\lambda}_n v) \approx \bar{\lambda}_n v \approx \sin(\bar{\lambda}_n v)$ from which Eq. (33) gives only the solution

$$\bar{\lambda}^2 = -\frac{\bar{\varpi}^2}{\bar{g}v}, \quad \Lambda = v, \tag{46}$$

which when substituted into Eq. (28) yields

$$\frac{(\bar{c}^2 + \bar{g}v)\bar{\varpi}^2}{\bar{c}^2 \bar{g}v} = \bar{\varpi}^2/\bar{\upsilon}^2 = \bar{k}^2, \quad \bar{\upsilon} = \sqrt{\frac{\bar{g}v}{1 + \bar{g}v/\bar{c}^2}}. \tag{47}$$

For a further simplification, we assume that there is no concentrated mass m_0 and moment I_0 of inertia at the end of the structure as well as retaining only the first natural mode $\phi_1(\xi)$ of the dry structure, which gives

$$\bar{\Omega} = 3.52, \quad \Psi_1 = \int_0^v \phi_1(\xi)\cos(\bar{\lambda}\xi)\,\mathrm{d}\xi. \tag{48}$$

As a result of this, Eq. (42) now takes the form

$$\begin{vmatrix} (\bar{\Omega}^2/\bar{\varpi}^2 - 1) & \Psi_1 \\ \Psi_1 & -\mathrm{i}\bar{k}v/\gamma \end{vmatrix} = 0. \tag{49}$$

The characteristic equation of the system is

$$\hat{\bar{\varpi}}^2 + 2\mathrm{i}\hat{\bar{\varpi}}\eta - 1 = 0, \quad \hat{\bar{\varpi}} = \bar{\varpi}/\bar{\Omega}, \quad \eta = \frac{\gamma(\Psi_1^2/v)}{2\bar{\Omega}}\bar{\upsilon}, \tag{50}$$

which has a conjugate complex solution

$$\hat{\bar{\varpi}} = -\mathrm{i}\eta \pm \sqrt{1 - \eta^2}. \tag{51}$$

The corresponding mode shape or eigenvector is obtained from Eq. (41) ($Q = 1$ for normalization):

$$Q = 1, \quad P = -\mathrm{i}\hat{\bar{\varpi}}\bar{\Omega}\gamma(\Psi_1/v)\bar{\upsilon}. \tag{52}$$

This example confirms the conclusions developed in Section 4. The parameter η plays a role of damping factor and the parameter $\bar{\upsilon}$ represents the *speed of radiation wave* influenced by both the free surface and the pressure waves. To further validate the theory developed in this paper, we discuss this solution as follows:

(i) *No free surface waves*: If free surface waves are not considered, the gravity acceleration tends to infinity, so that the speed of radiation wave $\bar{\upsilon} \to \bar{c}$ and therefore $\bar{k} = \bar{\kappa} = \bar{\varpi}/\bar{c}$ which is the case which considered only the pressure wave studied by Sommerfeld [1,2].

(ii) *Incompressible water*: Assume that the water is incompressible and the speed $\bar{c}$ of sound tends to infinity. As result of this, the speed of radiation wave $\bar{\upsilon} \to \sqrt{\bar{g}v}$ and $\bar{k} = \bar{\varpi}/\sqrt{\bar{g}v}$, where only the free surface wave is considered.

(iii) *Damping factor*: Eq. (50) shows that the damping factor of the system is proportional to the ratio γ of mass density, the speed $\bar{\upsilon}$ of radiation as well as a coupling term Ψ^2/v, which is physically explained as follows:
   (a) A large ratio γ of mass density implies a large mass density of water and therefore a large mount of energy is transmitted through the water to its infinite boundary and then dissipated, so that the damping factor is larger.
   (b) Increasing the speed of radiation $\bar{\upsilon}$ implies that the energy faster dissipates from the radiation boundary of the water, which causes a larger damping factor η.
   (c) The coupling term Ψ_1^2/v involves the integration in Eq. (48) which is influenced by the mode shape of the dry structure and the mode shape of the water as well as the integration on the wet surface of the

structure. As an estimation of the integration in Eq. (48), we have

$$\Psi_1 = \int_0^v \phi_1(\xi)\cos(\bar{\lambda}\xi)\,\mathrm{d}\xi = \phi_1(\tilde{\xi})\cos(\bar{\lambda}\tilde{\xi})v, \quad \Psi_1^2/v = \phi_1^2(\tilde{\xi})\cos^2(\bar{\lambda}\tilde{\xi})v, \tag{53}$$

where $\tilde{\xi}$ is a value between o and v. Therefore, a large depth v of water and the large integration value of Ψ_1 represent more strong coupling between the structure and the water, so that more mechanical energy of the structure transfers into the water and is then dissipated from the radiation boundary, therefore the damping factor is larger.

5.2. Deep water

As a reverse case of the shallow water case studied in Section 5.1, here we consider the deep water case in which the water depth tends to infinity. For convenience to estimate the values of the related functions, we introduce a parameter $\mathrm{i}\tilde{\lambda}_n = \bar{\lambda}_n$, so that Eq. (33) now takes a form

$$\tilde{\lambda}_n \tanh(\tilde{\lambda}_n v) = \frac{\bar{\varpi}^2}{\bar{g}}, \tag{54}$$

which has only one solution of $\tilde{\lambda}_n$ for a particular value $\bar{\varpi}$ as demonstrated by Xing et al. [15]. For this deep water case, this solution reduces to

$$\tilde{\lambda} = \frac{\bar{\varpi}^2}{\bar{g}}, \tag{55}$$

which when substituted into Eq. (28) yields

$$\bar{k}^2 = \frac{\bar{\varpi}^4}{\bar{g}^2} + \frac{\bar{\varpi}^2}{\bar{c}^2} = \frac{\bar{\varpi}^2}{\bar{v}^2}, \quad \bar{v} = \frac{\bar{g}\bar{c}}{\sqrt{\bar{g}^2 + \bar{c}^2\bar{\varpi}^2}}. \tag{56}$$

From Eqs. (33) and (38), it follows

$$\Lambda = \frac{v}{2} + \frac{\sinh(2\tilde{\lambda}v)}{4\tilde{\lambda}}, \quad \Psi_I = \int_0^v \phi_I(\xi)\cosh(\tilde{\lambda}\xi)\,\mathrm{d}\xi. \tag{57}$$

We retain only the first natural mode of the dry structure, and therefore Eq. (42) reduces to

$$\begin{vmatrix} (\bar{\Omega}^2/\bar{\varpi}^2 - 1) & \Psi_1 \\ \Psi_1 & -\mathrm{i}\bar{k}\Lambda/\gamma \end{vmatrix} = 0. \tag{58}$$

The characteristic equation of the system is

$$\hat{\bar{\varpi}}^2 + 2\mathrm{i}\hat{\bar{\varpi}}\eta - 1 = 0, \quad \hat{\bar{\varpi}} = \bar{\varpi}/\bar{\Omega}, \quad \eta = \frac{\gamma(\Psi^2/\Lambda)}{2\bar{\Omega}}\bar{v}, \tag{59}$$

which has a conjugate complex solution and the corresponding natural mode as follows:

$$\hat{\bar{\varpi}} = -\mathrm{i}\eta \pm \sqrt{1-\eta^2}, \tag{60}$$

$$Q = 1, \quad P = -\mathrm{i}\hat{\bar{\varpi}}\bar{\Omega}\gamma(\Psi_1/\Lambda)\bar{v}. \tag{61}$$

Compared with the case of the shallow water, the main differences are the different damping parameter η defined in Eq. (59) and the speed $\bar{v}$ of *radiation wave* given by Eq. (56). As discussed for the shallow water case, this speed of radiation wave reduces to the speed $\bar{c}$ of the pressure wave if *no free surface waves* ($\bar{g} \to \infty$) are considered, but to $\bar{g}/\bar{\varpi}$ if *incompressible water* ($\bar{c} \to \infty$) is considered.

5.3. No free surface wave considered

If no free surface wave is considered, the acceleration of gravity tends to infinity. As a result of this, Eq. (33) gives

$$\bar{\lambda}_0 = 0, \quad \Lambda_0 = v, \quad \bar{\lambda}_n = n\pi/v, \quad \Lambda_n = v/2, \quad n = 1, 2, 3, \ldots \tag{62}$$

which are real numbers and independent of the natural frequency $\bar{\omega}$ of the system. From Eq. (28), the corresponding parameters $\bar{k}_n$ is obtained as

$$\bar{k}_n^2 = \bar{\omega}^2/\bar{c}^2 - \bar{\lambda}_n^2, \quad n = 0, 1, 2, 3 \ldots \tag{63}$$

As an approximation, we choose only the first natural mode $\phi_1(\xi)$ of frequency $\bar{\Omega} = 3.52$ of the dry structure and consider the following two cases.

5.3.1. Considering only $\bar{Y}_0 = \cos(\bar{\lambda}_0 \xi) = 1$

In this case, we have

$$\Psi_{10} = \int_0^v \phi_1(\xi)\cos(\bar{\lambda}_0\xi)\,\mathrm{d}\xi = \int_0^v \phi_1(\xi)\,\mathrm{d}\xi, \quad \bar{k}_0^2 = \bar{\omega}^2/\bar{c}^2, \quad \Lambda_0 = v. \tag{64}$$

Eq. (42) now takes the form

$$\begin{vmatrix} (\bar{\Omega}^2/\bar{\omega}^2 - 1) & \Psi_{10} \\ \Psi_{10} & -\mathrm{i}\bar{k}_0\Lambda_0/\gamma \end{vmatrix} = 0, \tag{65}$$

which gives an equation same as Eq. (50) except the damping factor now takes the form

$$\eta = \frac{\gamma(\Psi_{10}^2/v)}{2\bar{\Omega}}\bar{c}, \tag{66}$$

in which the speed of radiation wave $\bar{v} = \bar{c}$ since no free surface waves are considered.

5.3.2. Considering $\bar{Y}_0 = \cos(\bar{\lambda}_0\xi) = 1$ and $\bar{Y}_1 = \cos(\bar{\lambda}_1\xi)$

For this case, we have

$$\Psi_{10} = \int_0^v \phi_1(\xi)\,\mathrm{d}\xi, \quad \Psi_{11} = \int_0^v \phi_1(\xi)\cos(\bar{\lambda}_1\xi)\,\mathrm{d}\xi, \quad \Lambda_0 = v, \quad \Lambda_1 = v/2,$$
$$\bar{k}_0^2 = \bar{\omega}^2/\bar{c}^2, \quad \bar{k}_1^2 = \bar{\omega}^2/\bar{c}^2 - \pi^2/v^2 = (\bar{\omega}^2/\bar{c}^2)\left[1 - \left(\frac{\pi\bar{c}}{\bar{\omega}v}\right)^2\right], \tag{67}$$

and Eq. (42) now takes the form

$$\begin{vmatrix} \bar{\Omega}^2/\bar{\omega}^2 - 1 & \Psi_{10} & \Psi_{11} \\ \Psi_{10} & -\mathrm{i}\bar{k}_0\Lambda_0/\gamma & 0 \\ \Psi_{11} & 0 & -\mathrm{i}\bar{k}_1\Lambda_1/\gamma \end{vmatrix} = 0. \tag{68}$$

which also gives an equation with the same structure as Eq. (50), that is,

$$\hat{\bar{\omega}}^2 + 2\mathrm{i}\hat{\bar{\omega}}\eta - 1 = 0, \quad \hat{\bar{\omega}} = \bar{\omega}/\bar{\Omega}, \quad \eta = \frac{\gamma(\Psi^2/v)\mu}{2\bar{\Omega}}\bar{c}, \quad \mu = \left\{1 + \frac{\Psi_{11}^2/(\bar{k}_1\Lambda_1)}{\Psi_{10}^2/(\bar{k}_0\Lambda_0)}\right\}. \tag{69}$$

in which a new parameter μ is introduced. This introduced parameter provides the influence of the second mode added. Actually, the parameter μ involves the natural frequency $\bar{\omega}$ to be determined. Since the ratio

$$\bar{k}_1/\bar{k}_0 = \sqrt{1 - \left(\frac{\pi^2}{v^2}\right)\Big/\left(\frac{\bar{\omega}^2}{\bar{c}^2}\right)}, \tag{70}$$

the parameter μ involves $\bar{\omega}^{-1}$ which does not increase the power of the natural frequency $\bar{\omega}$ of Eq. (68). Therefore, although the two fluid pressure functions $\bar{Y}_0 = \cos(\bar{\lambda}_0\xi) = 1$ and $\bar{Y}_1 = \cos(\bar{\lambda}_1\xi)$ are considered, we can obtain only one conjugate complex frequency due to only one degree of freedom of the structure is considered.

5.4. Incompressible water

For the case of incompressible water, the speed of sound in the water tends to infinity, which when substituted into Eq. (25) yields a non-dimensional relation

$$\bar{k}_n^2 + \bar{\lambda}_n^2 = 0, \quad n = 1, 2, 3, \ldots \tag{71}$$

and therefore we have

$$\bar{k}_n = \tilde{\lambda}_n, \quad n = 1, 2, 3, \ldots \tag{72}$$

where $\bar{\lambda}_n = \mathrm{i}\tilde{\lambda}_n$ and Eq. (64) are introduced as used for the case of deep water. As mentioned in Section 5.2, Eq. (54) has only one solution for a particular value of $\bar{\omega}$. We represent this solution in the form

$$\tilde{\lambda} = \frac{\bar{\omega}}{\bar{v}}, \tag{73}$$

where $\bar{v}$ is a speed of radiation wave introduced by us, which depends on the solution of Eq. (54) for a given problem. The corresponding function $\bar{Y} = \cosh(\tilde{\lambda}\xi)$ and the related integrations are the same as given by Eq. (57) except now the solution $\tilde{\lambda}$ is defined by Eq. (73).

We still consider the first natural mode of the dry structure, and therefore Eq. (42) yields the same form of the results as the ones given by Eqs. (58)–(61) except different definitions described herein, such as Eq. (73). We neglect these formulations and discussions.

6. Conclusions

The two-dimensional slender structure–water interaction system subject to a Sommerfeld radiation condition at infinity boundary of the water is investigated in this paper. The governing equations describing the structure–water interaction system are derived. To reveal the influences of the pressure wave in the water domain and the free surface wave on the speed of radiation wave at the infinity boundary where a Sommerfeld condition is imposed, a speed of radiation wave is introduced which is an extension of the original Sommerfeld condition. This new introduced speed of radiation wave reduces to the speed of the pressure wave if the free surface wave is not considered, but to the speed of the free surface wave if the water is considered as incompressible. The numerical formulations and the solution approach are developed. The theoretical analysis on the characteristics of the natural vibrations of the system is presented. Four selected case studies of the shallow water, deep water, no free surface wave and incompressible water are solved as examples to illustrate and to validate the theoretical analysis and solution method. The theoretical demonstrations and the examples confirm the following conclusions:

(i) The natural vibration of a two-dimensional slender structure–water interaction system subject to a Sommerfeld radiation condition at the infinity boundary of the water behaves as free damped vibration although there is no material damping in either solid or fluid. The damping is caused by the Sommerfeld radiation condition at infinite boundary where the energy of the system transmits from inside to outside.
(ii) The natural vibrations of the structure–water interaction system studied herein are governed by a complex eigenvalue problem which only has complex conjugate eigenvalues. The number of complex conjugate eigenvalues of the system equals the number of degrees of freedom of the dry structure in the system but remains independent of the fluid domain where the Sommerfeld condition is imposed.
(iii) For the dynamic response analysis of the structure–water interaction system subject to a Sommerfeld radiation condition, the natural modes of the dry structure are sufficient to construct a mode space to investigate any motions of the structure in the system since the number of the natural modes of the

structure–water system subject to a Sommerfeld condition is independent of the infinite water domain. It is not necessary to find the wet modes for the dynamic response analysis of the system studied in this paper.

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